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A model of amplification

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Abstract

A linear model showing behavior of the boundary layer type is developed as a generalization of such a matrix model of Bohl and Lancaster (1993). The governing equations have the form $\dot{x}(t) = Bx(t)$, $x(t) \geqslant 0$, $t \geqslant 0$, where the operator B, being the infinitesimal generator of a semigroup of operators of class $\mathscr{C}_0$, depends continuously on a parameter and has discontinuity in the dimension of the null space of B. It is shown that a singular behavior can be detected by a single parameter of imperfection. The analysis depends on a regularization technique for linear operator systems and on a perturbation result of spectral generalized inverse (Drazin). A partial order of the appropriate Banach spaces is another important tool in the investigation.

Keywords: $\mathscr{K}$-positive semigroups of operators; Perron eigenprojection; $\mathscr{K}$-cooperative systems

1. Introduction

The basis of this work is a reaction network given by Stryer [14, 15] in 1986 which solves the problem of how a (possibly very small) stimulus of light finds a chemical answer in our eye which is sent to the brain. This so-called cascade of vision involves an amplification effect which was mathematically studied in a series of papers [2–6]. The basic circuit results in a complicated system of differential equations which can be reduced to a smaller system of the form

$$\begin{aligned} &\dot{x}(t) = B(u)x(t) - \beta f g^{\mathrm{T}} x(t), \quad x(0) \geqslant 0, \\ &B(u) \in \mathbb{R}^{N,N}, \quad f, g \in \mathbb{R}^N, \quad u \geqslant 0, \quad \beta > 0. \end{aligned} \tag{1.1}$$

Here, the positive parameter u models the incoming stimulus: $u = 0$ means no outside stimulus and $u > 0$ signals a (possibly very small) stimulus of incoming light. It turns out that the positive

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parameter β in relation to u functions as a parameter of imperfection needed to detect the amplification measured in the cascade of vision. This basic concept has found some stages of generalization gradually revealing the true kernel of the abstract structure constructing this effect. We feel that in this paper we have arrived at a stage of development where the theory of nonnegative semigroups presents itself as the basic abstract concept and allows for very general applications in completely different fields. In contrast to the previous papers, we have no particular conditions on the null space of $B(u)$.

The organization of this paper is as follows. After some notations in Section 2 we develop the theory of cooperative systems in Section 3, which is the core of the stage of generalization presented in this paper. Sections 4, 5 and 6 show that Section 3 has developed a basis which allows all conclusions of the previous publications in our abstract setting.

2. Definitions, notation and preliminaries

Let $\mathscr{E}$ be a Banach space over the reals generated by a closed normal cone $\mathscr{K}$ [9], see also [10]. Let $\mathscr{E}'$ denote its dual and $\boldsymbol{B}(\mathscr{E})$ the space of bounded linear operators mapping $\mathscr{E}$ into $\mathscr{E}$. These latter spaces are assumed to be equipped by standard norms, so they are Banach spaces.

Let B be a linear operator mapping its domain $\mathscr{D}(B)$ into $\mathscr{E}$. We assume that $\mathscr{D}(B)$ is dense in $\mathscr{E}$. Then a dual map of B is defined and is denoted by B'. Let us recall that

$$B'v' = u'$$

is equivalent with the relations

$$\langle Bx, v' \rangle = \langle x, u' \rangle \quad \text{for all } x \in \mathscr{D}(B),$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathscr{E}$ and $\mathscr{E}'$.

Let $\mathscr{F}$ denote the *complex extension* of $\mathscr{E}$, i.e., $\mathscr{F} = \mathscr{E} \oplus \mathrm{i}\mathscr{E}$, with the norm

$$\| z \|_{\mathscr{F}} = \sup\{ \| x \cos\theta + y \sin\theta \|_{\mathscr{E}} \colon 0 \leqslant \theta \leqslant 2\pi \},$$

where $z = x + \mathrm{i}y$, $x, y \in \mathscr{E}$.

Let $\mathscr{K} \subset \mathscr{E}$ be a closed *normal and generating cone* [9], i.e., $\mathscr{K}$ satisfies the relations (a)–(f), where (a) $\mathscr{K} + \mathscr{K} \subset \mathscr{K}$, (b) $a\mathscr{K} \subset \mathscr{K}$, (c) $\mathscr{K} \cap (-\mathscr{K}) = \{0\}$, (d) $\bar{\mathscr{K}} = \mathscr{K}$ ($\bar{\mathscr{K}}$ denotes the norm closure of $\mathscr{K}$), (e) $\mathscr{E} = \mathscr{K} - \mathscr{K}$, i.e., for every $y \in \mathscr{E}$, there exist $y_j \in \mathscr{K}$, $j = 1, 2$, such that we have that $y = y_1 - y_2$, and (f) there exists a real $\delta > 0$ such that $\| x + y \|_{\mathscr{E}} \geqslant \delta \| x \|_{\mathscr{E}}$ whenever $x, y \in \mathscr{K}$.

Let $\mathscr{K}'$ be the *dual cone*, i.e.,

$$\mathscr{K}' = \{ x' \in \mathscr{E}' \colon \forall x \in \mathscr{K} \Rightarrow \langle x, x' \rangle \equiv x'(x) \geqslant 0 \}$$

and $\mathscr{K}^d$ the *d-interior* defined as

$$\mathscr{K}^d = \{ x \in \mathscr{K} \colon \forall x' \in \mathscr{K}',\ x' \neq 0 \Rightarrow \langle x, x' \rangle > 0 \}.$$

Obviously, if $\operatorname{Int} \mathscr{K} \neq \emptyset$, then $\operatorname{Int} \mathscr{K} = \mathscr{K}^d$. A closed normal cone $\mathscr{K}$ having $\operatorname{Int} \mathscr{K} \neq \emptyset$ is called *solid*.

A partial order is introduced into $\mathscr{E}$ by setting

$$x \leqslant y \text{ (or equivalently } y \geqslant x) \Leftrightarrow (y - x) \in \mathscr{K}.$$

An operator $T \in B(\mathscr{E})$ is called *$\mathscr{K}$-nonnegative* [9] if $T\mathscr{K} \subset \mathscr{K}$. A $\mathscr{K}$-nonnegative operator T is called *$\mathscr{K}$-irreducible* [13] if, for every pair $x \in \mathscr{K}$, $x \neq 0$, $x' \in \mathscr{K}'$, $x' \neq 0$, there is an index $p = p(x, x') \geqslant 1$, such that $\langle T^p x, x' \rangle > 0$. A $\mathscr{K}$-irreducible operator T is called *$\mathscr{K}$-primitive* if, for every $x \in \mathscr{K}$, $x \neq 0$, there exists a positive integer $p = p(x)$ such that $T^k x \in \mathscr{K}^d$ for all $k \geqslant p$.

Let Q be a densely defined linear operator mapping $\mathscr{E}$ into $\mathscr{E}$. An operator $T \in \boldsymbol{B}(\mathscr{E})$ is called *$\mathscr{K} - Q$-pseudoprimitive* if T is $\mathscr{K}$-primitive on $\mathscr{E} \ominus \operatorname{Ker} Q$.

Let $S, T \in B(\mathscr{E})$. We let

$$T \geqslant S \text{ (or equivalently } S \leqslant T) \Leftrightarrow (T - S)\mathscr{K} \subset \mathscr{K}.$$

Let $T \in B(\mathscr{E})$. By $\tilde{T}$ we denote the *complex extension* of T, i.e., $\tilde{T}z = Tx + \mathrm{i}Ty$, where $z = x + \mathrm{i}y$, $x, y \in \mathscr{E}$.

Let I denote the identity operator. Let $T \in B(\mathscr{E})$ and $\tilde{T}$ be its complex extension. The set

$$\rho(T) = \{\lambda \in \mathscr{C}: (\lambda I - \tilde{T})^{-1} \in \boldsymbol{B}(\mathscr{F})\}$$

is called the *resolvent set* of T. Its complement

$$\sigma(\tilde{T}) = \mathscr{C} \backslash \rho(\tilde{T})$$

is called the *spectrum* of T. We put $\sigma(T) := \sigma(\tilde{T})$.

The quantity

$$r(T) = \max\{|\lambda|: \lambda \in \sigma(T)\}$$

is called the *spectral radius* of T.

We define the *peripheral spectrum* of T by setting

$$\sigma_\pi(T) = \{\lambda \in \sigma(T): |\lambda| = r(T)\}.$$

Let $\tilde{T} \in B(\mathscr{F})$ and let $\mu \in \sigma(\tilde{T})$ be isolated. Then [16, pp. 305–306]

$$(\lambda I - \tilde{T})^{-1} = \sum_{k=0}^{\infty} A_k(\mu)(\lambda - \mu)^k + \sum_{k=1}^{\infty} B_k(\mu)(\lambda - \mu)^{-k},$$

where the $A_k(\mu), B_{k+1}(\mu) \in B(\mathscr{F})$, $k = 0, 1, \ldots$ and the following relations hold:

$$[B_1(\mu)]^2 = B_1(\mu)$$

and

$$B_{k+1}(\mu) = (\tilde{T} - \mu I)B_k(\mu).$$

In particular, if there is an index $q = q(\mu) < +\infty$, such that $B_k(\mu) = 0$ for $k > q$, the singularity μ is called *a pole* of the resolvent operator $(\lambda I - \tilde{T})^{-1}$ and $q = q(\mu)$ is called its *multiplicity* or *an order*. We also call the quantity q *the index* of $(\lambda I - \tilde{T})^{-1}$ and denote it by $\operatorname{ind}(\lambda I - \tilde{T})$. We write $q(\mu) = 0$ if $\mu \notin \sigma(T)$, i.e., if $\mu \in \rho(T)$ is a *regular point* of the resolvent operator.

An operator $\tilde{T} \in B(\mathscr{F})$ is said to have *property* "p" if the peripheral spectrum $\pi_\sigma(\tilde{T})$ consists of poles of the resolvent operator. We say that $T \in B(\mathscr{E})$ has the property "p" if its complex extension posesses this property. Similarly we say that an operator $T \in B(\mathscr{E})$ has a certain property if its complex extension $\tilde{T}$ possesses this property.

Lemma 2.1. *Let $T \in \boldsymbol{B}(\mathscr{E})$, $T\mathscr{K} \subset \mathscr{K}$ and let T have the property* "p". *Let $\hat{v}' \in \mathscr{K}'$ be strictly positive and such that*

$$T'\hat{v}' \leqslant \alpha'\hat{v}' \quad \textit{for some } \alpha' > 0. \tag{2.1}$$

Then

$$r(T) \leqslant \alpha'. \tag{2.2}$$

If

$$r(T) = \alpha',$$

then

$$\operatorname{ind}(\alpha' I - T)' = \operatorname{ind}(\alpha' I - T) = 1.$$

Proof. Let x_0 be an eigenvector corresponding to $r(T)$ (see [10]):

$$Tx_0 = r(T)x_0, \quad x_0 \in \mathscr{K} \setminus \{0\}.$$

By (2.1),

$$r(T)\langle x_0, \hat{v}' \rangle = \langle Tx_0, \hat{v}' \rangle = \langle x_0, T'\hat{v}' \rangle \leqslant \alpha' \langle x_0, \hat{v}' \rangle,$$

and, since $\langle x_0, \hat{v}' \rangle > 0$, the validity of (2.2) follows.

Let $r(T) = \alpha'$ and let $1 < q = \operatorname{ind}(\alpha' I - T)$. Set

$$S_N = \frac{1}{N} \sum_{k=1}^{N} k^{-q+1} \left(\frac{1}{r(T)} T \right)^k.$$

It is known [10] that

$$\lim_{N \to \infty} \| S_N - Z_q \| = 0,$$

where Z_q denotes the leading term in the main part of the Laurent expansion of $(\lambda I - T)^{-1}$ about $r(T)$.

$$(\lambda I - T)^{-1} = \sum_{k=0}^{\infty} A_k(\lambda - r(T))^k + \sum_{k=1}^{q} Z_k(\lambda - r(T))^{-k}.$$

From (2.1) it follows that

$$\frac{1}{N} \sum_{k=1}^{N} k^{-q+1} \left(\frac{1}{r(T)} T' \right)^k \hat{v}' \leqslant \frac{1}{N} \sum_{k=1}^{N} k^{-q+1} \hat{v}'. \tag{2.3}$$

Therefore, for every $x \in \mathscr{K}$,

$$\langle x, S_N' \hat{v}' \rangle \leqslant \frac{1}{N} \sum_{k=1}^{N} k^{-q+1} \langle x, \hat{v}' \rangle.$$

Since $Z_q \mathscr{K} \subset \mathscr{K}$ [10],

$$0 \leqslant \langle x, Z_q' \hat{v}' \rangle = \langle Z_q x, \hat{v}' \rangle \leqslant 0.$$

But this contradicts the fact that $0 \neq Z_q$ and $\hat{v}'$ is strictly positive. The proof of Lemma 2.1 is complete. □

Next we present also a "direct" version of Lemma 2.1. These two lemmas can be considered as mutually dual.

Lemma 2.2. *Let $T \in \mathscr{B}(\mathscr{E})$, $T\mathscr{K} \subset \mathscr{K}$ and let T have the property* "p". *Let $\hat{v} \in \mathscr{K}^d$ be such that*

$$T\hat{v} \leqslant \alpha \hat{v} \quad \textit{for some } \alpha > 0. \tag{2.4}$$

Then

$$r(T) \leqslant \alpha \tag{2.5}$$

and, if $\alpha = r(T)$, then

$$\operatorname{ind}(r(T)I - T) = 1.$$

Proof. Let $0 \neq \hat{x}_0' \in \mathscr{K}'$ be such that

$$T' \hat{x}_0' = r(T) \hat{x}_0'.$$

Note that the existence of such an element is guaranteed by the hypotheses of the Lemma (see [10]).

From (2.4) we deduce that

$$r(T) \langle \hat{v}, \hat{x}_0' \rangle \leqslant \alpha \langle \hat{v}, \hat{x}_0' \rangle,$$

and (2.5) then easily follows. It follows further from (2.4) that for $k = 1, 2, \ldots$

$$T^k \hat{v} \leqslant \alpha^k \hat{v},$$

or, equivalently,

$$\left(\frac{1}{\alpha} T \right)^k \hat{v} \leqslant \hat{v}. \tag{2.6}$$

Let $\alpha = r(T)$. If $\operatorname{ind}(r(T)I - T) = q > 1$, then (2.6) would imply that

$$\begin{aligned} S_N \hat{v} &= \frac{1}{N} \sum_{k=1}^{N} k^{-q+1} [r(T)]^{-k} T^k \hat{v} \\ &\leqslant \frac{1}{N} \sum_{k=1}^{N} k^{-q+1} \hat{v}. \end{aligned}$$

It follows that

$$0 \leqslant Z_q \hat{v} \leqslant 0.$$

However, $Z_q \neq 0$, $Z_q \mathscr{K} \subset \mathscr{K}$ [10] and $\hat{v} \in \mathscr{K}^d$, a contradiction.

Thus, $q = 1 = \operatorname{ind}((r(T)I - T)$. This completes the proof of Lemma 2.2. □

3. Cooperative systems

We consider a class of operators $B = B(u)$ assuming that

$$B(u) = B, \quad u \in \mathscr{U} = [0, u_0], \ u_0 > 0, \tag{3.1}$$

satisfy the following conditions (i)–(vii).

(i) The domains

$$\mathscr{D} = \mathscr{D}(B(u))$$

are dense in $\mathscr{E}$.

(ii) There exists $\hat{e}' \in \mathscr{K}' \subset \mathscr{E}'$, independent of u, such that

$$[B(u)]'\hat{e}' = 0 \quad \text{for all } u \in \mathscr{U}.$$

(iii) Each of the operators $B(u)$, $u \in \mathscr{U}$, is closed and its resolvent set is nonempty. Moreover, point $\hat{\lambda} = 0$ is an isolated singularity of the resolvent operators $R(\lambda, B(u)) = (\lambda I - B(u))^{-1}$.

(iv) Each of the operators $B(u)$, $u \in \mathscr{U}$, is the generator of a semigroup of operators $T(t; B(u)) \in \boldsymbol{B}(\mathscr{E})$ of class $\mathscr{C}_0$ with the following characteristics $M(B(u)) \in \mathscr{R}$ and $\omega(B(u)) \in \mathscr{R}$ such that [7, p. 360]

$$\| [(\lambda I - B(u))^{-1}]^k \| \leqslant M(B(u)) \frac{1}{|\lambda - \omega(B(u))|^k}, \quad \Re\lambda > \omega(B(u)).$$

It is assumed that

$$\| T(t; B(u)) \| \leqslant M(B(u)) \exp\{\omega(B(u))t\}, \quad u \in \mathscr{U}, \ t \geqslant 0,$$

and

$$T(t; B(u))\mathscr{K} \subset \mathscr{K} \quad \text{for } u \in \mathscr{U}, \ t \geqslant 0.$$

(v) Let n and N be positive integers. We let

$$\dim \operatorname{Ker} B(u) = n, \quad u \in \mathscr{U}, \ u \neq 0,$$

and require

$$\dim \operatorname{Ker} B(0) = N > n.$$

(vi) Each of the operators $T(t; B(u)) - I$, $t \geqslant 0$, $u \in \mathscr{U}$, has the property "p".

(vii) Let $p \geqslant 1$ be a positive integer. Let $\lambda_j(u)$, $j = 1, \ldots, p$, be isolated eigenvalues of $B(u)$ such that

$$0 \neq \lambda_j(u) \quad \text{for } u > 0,$$

$$\lim_{u \to 0_+} \lambda_j(u) = 0$$

and

$$\lambda \in \sigma(B(u)),\ \lambda \neq 0,\ \lambda \neq \lambda_j(u),\ j = 1, \dots, p, \Rightarrow |\lambda| \geqslant \rho_1,$$

where $\rho_1 > 0$ is u-independent.

It is assumed that hypotheses (i)–(vii) are fulfilled throughout the whole paper.

Proposition 3.1. *The semigroup-operator system has the following characteristics*:

$$\begin{aligned} &r(T(t;B(u))) = 1 \quad \text{for } t \geqslant 0,\ u \geqslant 0, \\ &\operatorname{ind}\{r[T(t;B(u))]I - T(t;B(u))\} = 1 \quad \text{for } t \geqslant 0,\ u \geqslant 0. \end{aligned} \tag{3.2}$$

Proof. We first refer to a well-known formula [7, p. 342]

$$R(\lambda, B(u))x = \int_0^{+\infty} T(t;B(u))\mathrm{e}^{-\lambda t}x\,\mathrm{d}t, \quad \lambda \geqslant \omega(B(u)),\ x \in \mathscr{E},$$

where

$$R(\lambda, B(u)) = (\lambda I - B(u))^{-1}.$$

According to hypothesis (ii), $\lambda\hat{e}' - B(u)\hat{e}' = \lambda\hat{e}'$, hence if λ is in the resolvent set,

$$\langle R(\lambda, B(u))x, \hat{e}'\rangle = \frac{1}{\lambda}\langle x, \hat{e}'\rangle \quad \text{for all } x \in \mathscr{E}. \tag{3.3}$$

Since formula (3.3) is valid for all $x \in \mathscr{E}$ and all λ from the resolvent set of $B(u)$, and since the latter set contains the whole half-plane $\{\lambda\colon \Re\lambda > 0\}$, it follows that the Laplace transform of $\langle x, [T(t;B(u))]'\hat{e}'\rangle$ coincides with the Laplace transform of the constant function assuming the value 1. Therefore,

$$\left\langle x, \int_0^{+\infty} [T(t;B(u))]'\hat{e}'\right\rangle = \left\langle x, \int_0^{+\infty} \mathrm{e}^{-\lambda t}\hat{e}'\right\rangle \mathrm{d}t = \frac{1}{\lambda}\langle x, \hat{e}'\rangle.$$

As a result we deduce that [17, p. 26]

$$[T(t;B(u))]'\hat{e}' = \hat{e}', \quad t \geqslant 0,\ u \geqslant 0.$$

Since, by (vii), each of the operators $T(t;B(u))$ has the property "p", the statement of Proposition 3.1 is then a consequence of Lemma 2.1. □

A consequence of Proposition 3.1 is

Corollary 3.2. *The infinitesimal generator $B(u)$ has the following properties*:

$$\sigma(B(u)) \subset \{\lambda\colon \Re\lambda \leqslant 0\} \cup \{0\}, \quad u \geqslant 0. \tag{3.4}$$

In the Laurent expansion

$$R(\lambda, B(u)) = \sum_{k=0}^{\infty} A_k(u)\lambda^{-k} + \sum_{k=1}^{\infty} B_{-k}(u)\lambda^{-k}, \tag{3.5}$$

which holds, according to (iii), *for* $\lambda \in \Gamma(\rho)$, *where*

$$\Gamma(\rho) = \{\lambda\colon 0 \neq |\lambda| \leqslant \rho,\ \rho > 0\},$$

there is only a single nonzero coefficient in the main part of (3.5) *and it coincides with the Perron eigenprojection* $P_0(u)$, i.e.,

$$B_{-k} = 0, \quad k = 2, 3, \ldots, B_{-1} = P_0(u) \qquad \text{and} \qquad P_0(u)\mathscr{K} \subset \mathscr{K}, \tag{3.6}$$

or equivalently

$$\operatorname{ind} B(u) = 1, \quad u \geqslant 0. \tag{3.7}$$

If the semigroup of operators $\{T(t; B(u))\}$ *is* $\mathscr{K} - B(u)$*-pseudoprimitive for* $u > 0$ *and for some* t, *then the infinitesimal generators* $B(u)$, $u \geqslant 0$, *have the following form*:

$$-I + B(u) = -P_0(u) + C(u), \tag{3.8}$$

where

$$P_0(u)C(u) = C(u)P_0(u), \quad u \geqslant 0, \tag{3.9}$$

$P_0(u)$, $u \in \mathscr{U}$, *is the spectral projection associated with the null space of* $B(u)$ [16, p. 299] *and* $C(u)$ *is the infinitesimal generator of a semigroup of operators* $T(t; C(u))$ *of class* $\mathscr{C}_0$ *such that*

$$\| T(t; C(u)) \| \leqslant M(C(u)) \exp\{-\omega(C(u))t\}, \quad \omega(C(u)) > 0, \tag{3.10}$$

for $t \geqslant 0$, $u > 0$.

Proof. The validity of (3.4) is obtained by applying formula [7, p. 457]

$$\exp[\xi\sigma(B(u))] \subset \sigma(T(\xi; B(u))). \tag{3.11}$$

Since $\exp\{0\} = 1$, we see that the eigenprojection onto the null space of $B(u)$ coincides with the eigenprojection of $T(t; B(u))$ corresponding to $1 = r(T(t; B(u)))$. The conclusion (3.6) follows according to the Frobenius theory of $\mathscr{K}$-positive operators [10] applied to $T(t; B(u))$. In more detail, since $r(T(t; B(u)))$ is a pole of the resolvent operator $R(\lambda, T(t; B(u)))$, so is 0 a pole of $R(\lambda, B(u))$. Relation (3.7) is implied by (3.2) because of (3.11).

Relations (3.8)–(3.10) follow from the fact that the peripheral spectrum $\sigma_\pi(T(t); B(u))$ is an isolated singleton [11]. □

Remark 3.3. In the above proof we have used the concept of the $\mathscr{K} - B(u)$-pseudoprimitivity of $T(t; B(u))$ according to our definition from Section 2 which is a slight modification of the definition given in [11]. It is easy to see that the result in [11] that we refer to remains valid for the modified case as well.

Proposition 3.4. *Let*

$$0 < \delta < \min\{|\lambda|\colon \lambda \in \sigma(B(u)) \backslash \{0\}\}$$

and

$$f(\lambda) = \begin{cases} 1/\lambda & \text{if } |\lambda| > \delta > 0, \\ 0 & \text{if } |\lambda| \leqslant \delta. \end{cases}$$

Then $B(u)^{\#} = f(B(u))$ *is the* generalized spectral inverse *of* $B(u)$, *i.e.,* $f(B(u))$ *satisfies the following three relations* (α–γ):

(α) $B(u)^{\#} B(u)x = B(u)B(u)^{\#} x, \quad \forall x \in \mathscr{D}(B(u))$,

(β) $B(u)^{\#} B(u)B(u)^{\#} x = B(u)^{\#} x, \quad \forall x \in \mathscr{D}(B(u))$,

(γ) $B(u)^{\#} B(u)^2 x = B(u)x, \quad \forall x \in \mathscr{D}(B(u))$.

Lemma 3.5. *The operator-function* $B^{\#}(u) = B(u)^{\#}$ *satisfies* $[B(u)^{\#}]'e' = 0$ *for all* $u \geqslant 0$. *Its singularity at* $u = 0$ *is of the type*

$$\mathrm{O}\left(\frac{1}{|\lambda(u)|^q}\right),$$

where $\lambda(u)$ *is an eigenvalue of* $B(u)$ *for which*

$$|\lambda(u)|^q \leqslant \min\{|\lambda_j(u)|^{q_j}: 0 \neq \lambda_j(u) \to 0, \text{ as } u \to 0, \lambda_j(u) \in \sigma(B(u))\},$$

where

$$q_j = \operatorname{ind}(\lambda_j(u)I - B(u)) \quad \text{and} \quad q = \max\{q_1, \ldots, q_p\}.$$

In particular, if $\mathscr{E}$ *is finite dimensional and* $B(u)$ *is a polynomial in* u *having* $u = 0$ *as root of order* $r \geqslant 1$, *then* $B^{\#}(u)$ *has a pole of order* rq *at* $u = 0$.

Proof. Let $\sigma(u)$ and $\sigma_1(u)$ be negative real numbers such that

$$0 \neq \lambda \in \sigma(B(u)) \Rightarrow \Re\lambda < \sigma(u),$$

$$\lambda \in \sigma(B(u)),\ \lambda \neq \lambda_j(u),\ j = 0, 1, \ldots, p,\ \lambda_0(u) = 0, \Rightarrow \Re\lambda < \sigma_1(u).$$

$$\mathscr{C}(u) = \{\lambda: \Re\lambda \leqslant \sigma(u)\}$$

and

$$\mathscr{C}_{C(u)} = \{\lambda: \Re\lambda \leqslant \sigma_1(u)\}.$$

Moreover, let

$$\mathscr{C}_j(u) = \{\lambda: |\lambda - \lambda_j| \leqslant \rho_j(u),\ \rho_j(u) > 0\}, \quad j = 1, \ldots, p,$$

be such that

$$\sigma(B(u)) \cap \mathscr{C}_j(u) = \{\lambda_j(u)\}.$$

Then [16, p. 293]

$$B(u)^{\#} = \frac{1}{2\pi \mathrm{i}} \int_{\partial\mathscr{C}(u)} \lambda^{-1}(\lambda I - B(u))^{-1}\, \mathrm{d}\lambda.$$

According to our hypotheses [16, p. 305]

$$B(u)^{\#} = \frac{1}{2\pi i}\int_{\partial\mathscr{C}_{C(u)}} \lambda^{-1}(\lambda I - B(u))^{-1}\,d\lambda$$
$$+ \sum_{j=1}^{p}\sum_{k=1}^{q_j}\frac{1}{[\lambda_j(u)]^k}[B(u)-\lambda_j(u)I]^{k-1}P_j(u), \tag{3.12}$$

where

$$P_j(u) = \frac{1}{2\pi i}\int_{\partial\mathscr{C}_j(u)}(\lambda I - B(u))^{-1}\,d\lambda. \tag{3.13}$$

Since the first summand in (3.12) is uniformly bounded with respect to u the assertion of Lemma 3.5 then easily follows. □

4. Perturbation of the spectral generalized inverse

Most of the results of this section are a kind of paraphrases of the results obtained in [5] for the case of operators on $\mathfrak{R}^n$.

Let us assume that A is a given infinitesimal generator of a semigroup of operators of class $\mathscr{C}_0$ and $V: \mathscr{D}(V) \to \mathscr{E}$ satisfies

$$\operatorname{Im} V \subset \operatorname{Im} A, \qquad \mathscr{D}(V)\cap\mathscr{D}(A) \quad \text{dense in } \mathscr{E} \tag{4.1}$$

We also assume that A is singular, i.e., $0 \in \sigma(A)$. Moreover, let

$$\operatorname{ind} A = 1, \tag{4.2}$$

i.e., $v = 0$ is a simple pole of the resolvent operator $(\lambda I - A)^{-1}$.

Theorem 4.1. *Let A be the infinitesimal generator of a semigroup of operators of class $\mathscr{C}_0$, let A be singular, i.e., $0 \in \sigma(A)$, and let* (4.2) *hold. Let V be any densely defined linear operator on $\mathscr{E}$ such that*

$$\operatorname{Ker} A \subset \mathscr{D}(V), \qquad \mathscr{E} = \operatorname{Im} A \oplus \operatorname{Ker} A$$

and (4.1) *holds. Furthermore, let $I + VA^{\#}$ be boundedly invertible. Then $(A+V)^{\#}$ exists and*

$$(A+V)^{\#} - A^{\#} = -A^{\#}(I + VA^{\#})^{-1}$$
$$\times\{VA^{\#} + A^{\#}(I+VA^{\#})^{-1}V(A^{\#}A - I)\}. \tag{4.3}$$

In particular, if $b \in \operatorname{Im} A$,

$$(A+V)^{\#}b - A^{\#}b = -A^{\#}(I+VA^{\#})^{-1}VA^{\#}b. \tag{4.4}$$

Proof. By hypothesis,

$$\mathscr{E} = \operatorname{Im} A \oplus \operatorname{Ker} A$$

is a direct decomposition of $\mathscr{E}$. Such decomposition is guaranteed by (4.2). Let

$$A|_{\operatorname{Im} A} = A_1$$

and, similarly, because of (4.1), let

$$V|_{\operatorname{Im} A} = V_{11}.$$

Furthermore, let V_{12} be the restriction of V on $\operatorname{Ker} A$ with the values in $\mathscr{E}$.

Formally, we can write A and $A + V$ as follows:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$A + V = \begin{pmatrix} A_1 + V_{11} & V_{12} \\ 0 & 0 \end{pmatrix},$$

respectively. It follows that

$$A^{\#} = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.5}$$

and

$$I + VA^{\#} = \begin{pmatrix} I_{\operatorname{Im} A} + V_{11}A_1^{-1} & 0 \\ 0 & I_{\operatorname{Ker} A} \end{pmatrix}. \tag{4.6}$$

Since $I + VA^{\#}$ is boundedly invertible, it follows from (4.6) that $I_{\operatorname{Im} A} + V_{11}A_1^{-1}$ is boundedly invertible too. It is then easy to see that

$$(A + V)^{\#} = \begin{pmatrix} (A_1 + V_{11})^{-1} & (A_1 + V_{11})^{-2}V_{12} \\ 0 & 0 \end{pmatrix}. \tag{4.7}$$

Since

$$\begin{aligned} (A_1 + V_{11})^{-1} - A_1^{-1} &= -(A_1 + V_{11})^{-1}V_{11}A_1^{-1} \\ &= -A_1^{-1}(I_{\operatorname{Im} A} + V_{11}A_1^{-1})^{-1}V_{11}A_1^{-1}, \end{aligned}$$

Eqs. (4.7) and (4.5) give

$$\begin{aligned} &(A + V)^{\#} - A^{\#} \\ &= \begin{pmatrix} -A_1^{-1}(I + V_{11}A_1^{-1})^{-1}V_{11}A_1^{-1} & (A_1 + V_{11})^{-2}V_{12} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -A_1^{-1}(I_{\operatorname{Im} A} + V_{11}A_1^{-1})^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad \times \left[\begin{pmatrix} V_{11}A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -A_1^{-1}(I_{\operatorname{Im} A} + V_{11}A_1^{-1})^{-1}V_{12} \\ 0 & 0 \end{pmatrix} \right]. \end{aligned} \tag{4.8}$$

In the last operator-matrix use

$$V(A^{\#}A - I) = \begin{pmatrix} 0 & -V_{12} \\ 0 & 0 \end{pmatrix}$$

and (4.8) yields (4.3).

If $b \in \operatorname{Im} A$, then $(A^{\#}A - I)b = 0$ as an easy consequence of the definition of $A^{\#}$ and (4.4) follows from (4.3). $\square$

Remark 4.2. This theorem generalizes a result of Bohl [2–4] and also of Bohl and Lancaster [5]. It has already been pointed out in these papers that the classical matrix Sherman–Morrison–Woodbury formula is a special case.

Proposition 4.3. *Let A be a densely defined linear operator mapping $\mathscr{D}(A) \subset \mathscr{E}$ into $\mathscr{E}$. Let $0 \in \sigma(A)$ be a simple pole of its resolvent operator $(\lambda I - A)^{-1}$. Furthermore, let V be another densely defined linear operator mapping $\mathscr{D}(V)$ into $\mathscr{E}$ with*

$$\operatorname{Ker} A \subset \mathscr{D}(V), \qquad \mathscr{D}(V) \cap \mathscr{D}(A) \quad \text{dense in } \mathscr{E} \tag{4.9}$$

and

$$\operatorname{Im} V \subset \operatorname{Im} A. \tag{4.10}$$

Then

$$W = I - A^{\#}(I + VA^{\#})^{-1}V \tag{4.11}$$

is boundedly invertible on $\mathscr{E}$.

Furthermore,

$$(A + V)W = A \tag{4.12}$$

and

$$W^{-1} = I + A^{\#}V. \tag{4.13}$$

Proof. It is a matter of easy calculation to show that

$$(I + A^{\#}V)W = I$$

and

$$W(I + A^{\#}V) = I.$$

Indeed,

$$\begin{aligned}(I + A^{\#}V)&[I - A^{\#}(I + VA^{\#})^{-1}V] \\ &= I + A^{\#}V - (I + A^{\#}V)A^{\#}(I + VA^{\#})^{-1}V \\ &= I + A^{\#}V - A^{\#}(I + VA^{\#})(I + VA^{\#})^{-1}V \\ &= I + A^{\#}V - A^{\#}V = I,\end{aligned}$$

and, similarly,

$$\begin{aligned} &W(I + A^{\#} V) \\ &\quad = I + A^{\#} V - A^{\#}(I + VA^{\#})^{-1}(I + VA^{\#})V \\ &\quad = I + A^{\#} V - A^{\#} V = I. \end{aligned}$$

Finally,

$$\begin{aligned} &(A + V)W \\ &\quad = (A + V)[I - A^{\#}(I + VA^{\#})^{-1} V] \\ &\quad = A + V - AA^{\#}(I + VA^{\#})^{-1} V - VA^{\#}(I + VA^{\#})^{-1} V \end{aligned}$$

(using the relation $AA^{\#} V = V$)

$$\begin{aligned} &\quad = A + V - (I + VA^{\#})^{-1} V - (I - I + VA^{\#})(I + VA^{\#})^{-1} V \\ &\quad = A + V - (I + VA^{\#})^{-1} V + (I + VA^{\#})^{-1} V - V = A. \qquad \square \end{aligned}$$

Corollary 4.4. *For any $b \in \operatorname{Im} A$, W determines a one-to-one map between the solution manifolds of $Ax = b$ and $(A + V)x = b$.*

Proof. It follows immediately from (4.12) and the fact that W is boundedly invertible. $\square$

Remark 4.5. In particular, note that W maps $\operatorname{Ker} A$ one-to-one onto $\operatorname{Ker}(A + V)$.

5. A regularization technique

Let A be a linear operator satisfying the conditions of the previous section. Let us define a regularization operator as follows:

$$B_0(u) = B(u) + P_0(u), \tag{5.1}$$

where $P_0(u)$ is defined in (3.6). We see that $0 \notin \sigma(B_0(u))$ for $u > 0$ and, thus, $B_0(u)$ is boundedly invertible for $u > 0$.

We should recall that

$$B(u)P_0(u)x = P_0(u)B(u)x \quad \text{for all } x \in \mathscr{D}(B(u)) \text{ and } u \in \mathscr{U}.$$

Thus the equation

$$B(u)x = b, \quad b \in \operatorname{Im} B(u),$$

has a solution of the form

$$x^* = B(u)^{\#} b + v,$$

with any $v = P_0(u)v$, $v \in \mathscr{E}$.

Indeed,

$$B(u)x^* = B(u)B(u)^{\#}b + B(u)v = B(u)B(u)^{\#}b = b.$$

In other words, x^* is a unique solution to

$$B_0(u)x = b + v.$$

Theorem 5.1. *Let $x = x(u;t)$, $x_\beta(u;t)$ be defined for $u > 0$ and $\beta \geqslant 0$ by*

$$\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}x = B(u)x, \quad x(u;0) = x_0 \in \mathscr{X},$$

and

$$\dot{x}_\beta = \frac{\mathrm{d}}{\mathrm{d}t}x_\beta = B(u)x_\beta + \beta V x_\beta, \quad x_\beta(u;0) = x_{\beta,0}(0) \in \mathscr{X}, \ \beta \geqslant 0,$$

respectively. Assume that

(α) *$B(u;\beta) = B(u) + \beta V$ defines a cooperative system, i.e., the conditions* (i)–(vii) *hold and, moreover, let*

(viii) $V'\hat{e}' = 0$.

(β) *$0 \in \sigma(B(u,\beta))$ is a simple pole of the resolvent operator $(\lambda I - B(u,\beta))^{-1}$ for $u \geqslant 0$ and $\beta \geqslant 0$.*

(γ) *$\mathscr{D}(B(u)) \cap \mathscr{D}(V)$ is dense in $\mathscr{E}$ and*

$$\operatorname{Im} V \subset \operatorname{Im} B(u), \quad \operatorname{Ker} B(u) \subset \mathscr{D}(V), \qquad u \geqslant 0,$$

holds together with

$$\dim \operatorname{Ker} B(0;0) = N,$$

$$N > n = \dim \operatorname{Ker} B(u;\beta) \quad \textit{for } (u,\beta) \neq (0,0).$$

Then for any $\beta \geqslant 0$ for which $I + \beta B(u)^{\#}$ is boundedly invertible and for $u > 0$

$$x_\beta(u;\infty) = (I - \beta B(u)^{\#}[I + \beta V B(u)^{\#}]^{-1} V)x(u;\infty). \tag{5.2}$$

Proof. The vectors $x(u;\infty)$ and $x_\beta(u;\infty)$ are defined for $u > 0$ and $\beta \geqslant 0$ by

$$B(u)x(u;\infty) = 0, \quad \langle x(u;\infty), \hat{e}' \rangle = 1, \tag{5.3}$$

and

$$[B(u) + \beta V]x_\beta(u;\infty) = 0, \quad \langle x_\beta(u;\infty), \hat{e}' \rangle = 1, \tag{5.4}$$

respectively.

We apply Corollary 4.4 with $B(u) \to A$, $\beta V \to V$, $b = 0$ to Eqs. (5.3) and (5.4). In this case (4.11) gives

$$W = I - \beta B(u)^{\#}[I + \beta V B(u)^{\#}]^{-1} V.$$

Hence, by Corollary 4.4,

$$z = \{I - \beta B(u)^{\#}[I + \beta V B(u)^{\#}]^{-1}\}x(u,\infty)$$

satisfies $[B(u) + \beta V]z = 0$. Furthermore, $[B(u)^{\#}]'\hat{e}' = 0$ for all $u > 0$ and so $\langle x(u;\infty), \hat{e}'\rangle = 1$ implies that $\langle z, \hat{e}'\rangle = 1$. Hence $z = x_\beta(u;\infty)$. $\square$

Remark 5.2. The β-problem in Theorem 5.1 is an abstract version of the cascade of vision problem (1.1) given in the Introduction.

6. Problem (P) and the boundary layer phenomenon

Problem (P). Determine the behavior of the steady-state solutions of the dynamical system

$$\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}x = B(u)x, \quad x(u;0) = x_0 \in \mathscr{X}, \tag{6.1}$$

and its perturbation

$$\dot{y} = \frac{\mathrm{d}}{\mathrm{d}t}y = B(u)y + \beta V y, \quad y(u;0) = y_0 \in \mathscr{X}, \quad \beta \geqslant 0, \tag{6.2}$$

where $\mathscr{U} = [0, u_0)$ is a parameter set, β is a perturbation parameter and V maps its dense domain $\mathscr{D}(V) \subset \mathscr{E}$, $\operatorname{Ker} B(u) \subset \mathscr{D}$ into $\mathscr{E}$, $\mathscr{D}(B(u)) \cap \mathscr{D}(V)$ dense in $\mathscr{E}$, and $\operatorname{Im} V \subset B(u)$, $u \geqslant 0$.

Remark. We are going to show that the solutions to the above systems happen to be singular in a vicinity of $u = 0$.

The steady-state solution $x(u;\infty)$ can be cast in the following form:

$$x(u;\infty) = \frac{P_0(u)v}{\langle P_0(u)v, \hat{e}'\rangle} = \frac{P_0(u)v}{\langle v, [P_0(u)]'\hat{e}'\rangle} = \frac{P_0(u)v}{\langle v, \hat{e}'\rangle},$$

where $v \in \mathscr{X}$ is such that

$$P_0(u)v \neq 0.$$

We see that $x = x(u;\infty)$ is continuous for $u > 0$.

By (vii) the spectrum of $B(u)$ contains elements $\lambda_j(u)$, $j = 1, \ldots, p$, such that

$$0 \neq \lambda_j(u) \quad \text{for } u > 0 \tag{6.3}$$

and

$$\lim_{u \to 0_+} \lambda_j(u) = 0. \tag{6.4}$$

Moreover, let

$$\lambda \in \sigma(B(u)),\ \lambda \neq 0,\ \lambda \neq \lambda_j(u),\ j = 1, \ldots, p, \Rightarrow |\lambda| > \rho_1, \tag{6.5}$$

where $\rho_1 > 0$ is u-independent. We set

$$P_{0p}(u) := P_0(u) + \sum_{j=1}^{p} P_j(u),$$

where $P_j(u)$ denotes the spectral projection onto the generalized eigenspace of $\lambda_j(u), j = 1, \ldots, p$ as shown in (3.13).

The above hypotheses imply that the projection $P_{0p}(u)$ is a continuous function of u within $[0, \tilde{u}_0)$ for some $\tilde{u}_0 > 0$. It follows that

$$x(u;\infty) = \frac{P_{0p}(u)v}{\langle P_{0p}(u)v, \hat{e}'\rangle} \tag{6.6}$$

with $v \in \mathscr{X}$ such that $P_{0p}(u)v \neq 0$ for $u > 0$. Thus, since $[P_j(u)]'\hat{e}' = 0, j = 1, \ldots, p$, for $u > 0$,

$$\lim_{u\to 0} x(u;\infty) = \begin{cases} \lim\limits_{u\to 0} \dfrac{P_{0p}(u)v}{\langle P_{0p}(u)v, \hat{e}'\rangle}, \\ \lim\limits_{u\to 0} \dfrac{P_{0p}(u)v}{\langle v, \hat{e}'\rangle} = x(0;\infty). \end{cases} \tag{6.7}$$

This result can be stated as (cf. [8, Theorem 5.1, p. 107])

Theorem 6.1. *Let $B(u)$ be analytic in u for $u > 0$. If $B(u)$ is such that hypotheses* (i)–(viii) *hold, then every steady-state solution $x(u;\infty)$, $\langle x(u;\infty), \hat{e}'\rangle = 1$, is continuous within $u \in [0, \tilde{u}_0)$, $\tilde{u}_0 > 0$.*

Next we are going to investigate some particular perturbations of our cooperative operator $B(u)$.

Theorem 6.2. *Given the hypotheses of Theorem* 5.1, *let V have a form*

$$V = FG^*, \quad F : \mathscr{E}_r' \to \mathscr{E}, \quad G^* : \mathscr{E} \to \mathscr{E}_r' \subset \mathscr{E}', \tag{6.8}$$

where $\mathscr{E}_r$ is an r-dimensional subspace of $\mathscr{E}$. Let

$$\operatorname{Ker} B(u) \subset \mathscr{D}(V),$$

and let

$$\operatorname{rank} V = r \geqslant 1. \tag{6.9}$$

Then

$$x_\beta(u;\infty) = \{I - \beta B(u)^{\#} F[I_r + \beta G^* B(u)^{\#} F]^{-1} G^*\} x(u;\infty). \tag{6.10}$$

Proof. The result follows from relation (5.2) if we can show that

$$[I + \beta V B(u)^{\#}]^{-1} F = F[I_r + \beta G^* B(u)^{\#} F]^{-1}. \tag{6.11}$$

To show (6.11) we first observe that

$$U = (I + \beta V B(u)^{\#})^{-1} F$$

can be written in the form

$$U = FM, \tag{6.12}$$

where M being $r \times r$ must be regular. We see that

$$\begin{aligned} F[I_r + \beta G^* B(u)^\# F]M &= FM + \beta FG^* B(u)^\# FM \\ &= FM + \beta VB(u)^\# FM \\ &= (I + \beta VB(u)^\#)FM \\ &= F, \end{aligned}$$

and this implies

$$FM = [I + \beta VB(u)^\#]^{-1} F \tag{6.13}$$

and

$$(I_r + \beta G^* B(u)^\# F)M = I_r.$$

This means that

$$M = [I_r + \beta G^* B(u)^\# F]^{-1}.$$

Substituting in (6.12), we obtain (6.11). □

Since we have

$$\operatorname{Im} F \subset \operatorname{Im} V \subset \operatorname{Im} B(u),$$

we know already that

$$B(u)^\# F = [B_0(u)]^{-1} F. \tag{6.14}$$

Consequently, $B(u)^\# F$ has the smoothness properties of $B(u)$ for $u > 0$. It follows from (6.10) that at each β for which (6.10) holds, $x_\beta(u; \infty)$ is a smooth function of u in the same sense as $B(u)$ for $u > 0$.

We consider the limiting behavior of $x_\beta(u; \infty)$ as $u \to 0$ and compare it with that of $x(u; \infty)$. For $\beta \geqslant 0$ write

$$x_\beta(0; \infty) = \lim_{u \to 0_+} x_\beta(u; \infty) \tag{6.15}$$

when the limit exists.

From Eq. (6.10) we have for $u > 0$ and $\beta > 0$

$$x_\beta(u; \infty) = [I - H(u; \beta)] x(u; \infty), \tag{6.16}$$

where

$$H(u; \beta) = \beta B(u)^\# F [I_r + \beta G^* B(u)^\# F]^{-1} G^*, \tag{6.17}$$

so first consider the existence of the limit

$$H_0 = \lim_{u \to 0_+} \beta B(u)^{\#} F [I_r + \beta G^* B(u)^{\#} F]^{-1} G^*. \tag{6.18}$$

Recalling Theorem 6.1 it is convenient at this point to assume an analytic dependence of $B(u)$ on u. This permits discussion of the possible singularity of $B(u)^{\#} F$ at $u = 0$. Note that in Eq. (5.1) we have

$$B(u)^{\#} F = [B_0(u)]^{-1} F \neq 0 \tag{6.19}$$

whenever $F \neq 0$. Furthermore, $[B_0(u)]^{-1}$ has a pole at $u = 0$. This is a consequence of the following hypothesis which we introduce now.

(ix) *The operator-function* $[B_0(u)]^{-1}$ *is meromorphic*, in particular, if 0 is a singularity of $[B_0(u)]^{-1}$, then it is a pole.

Therefore, for $0 < |u|$ and $|u|$ small enough we have the expansions

$$[B_0(u)]^{-1} = \sum_{k=-q}^{\infty} \tilde{A}_k u^k, \quad q \geqslant 1, \tag{6.20}$$

and

$$B_0(u) = \sum_{k=0}^{\infty} \tilde{B}_k u^k. \tag{6.21}$$

The assumption $\tilde{A}_{-k} = 0$ for $k = 1, \ldots, q$ implies that

$$\tilde{A}_0 \tilde{B}_0 = I,$$

but this is not possible, because, by hypothesis, $0 \in \sigma(B_0(0))$. We easily deduce that

$$\begin{gathered} B(u)^{\#} F = \sum_{k=-t}^{\infty} C_k u^k, \\ C_k = \tilde{A}_k F, \quad q \geqslant t \geqslant 1, \quad C_{-t} \neq 0, \end{gathered} \tag{6.22}$$

and, since $I_r + \beta G^* B(u)^{\#} F$ has meromorphic entries,

$$[I_r + \beta G^* B(u)^{\#} F]^{-1} = \sum_{k=-m}^{\infty} A_k u^k, \quad A_{-m} \neq 0, \tag{6.23}$$

where m is an integer.

It is not difficult to show that

$$H(u;\beta) = \beta \sum_{k=0}^{m+t} J_k G^* u^{k-(m+t)} + \mathrm{O}(u), \tag{6.24}$$

where

$$J = \begin{pmatrix} J_0 \\ J_1 \\ \vdots \\ J_{m+t} \end{pmatrix},$$

$$J = \begin{pmatrix} C_{-t} & 0 & \cdots & 0 & 0 \\ C_{-t+1} & C_{-t} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ C_m & C_{m-1} & \cdots & C_{-t+1} & C_{-t} \end{pmatrix} \begin{pmatrix} A_{-m} \\ A_{-m+1} \\ \vdots \\ A_t \end{pmatrix},$$

i.e.,

$$\begin{aligned} J_0 &= C_{-t}A_{-m}, \\ J_1 &= C_{-t+1}A_{-m} + C_{-t}A_{-m+1}, \\ &\vdots \\ J_{m+t} &= C_m A_{-m} + C_{m-1}A_{-m+1} + \cdots + C_{-t}A_t. \end{aligned}$$

Proposition 6.3. *The limit H_0, defined by* (6.18), *exists if and only if either*
(α) $m + t < 0$ *or*
(β) $m + t > 0$ *and* $J_k = 0$ *for* $k = 0, \ldots, m + t - 1$.
When $m + t < 0$, *then* $H_0 = 0$, *and when* $m + t \geqslant 0$, *then*

$$H_0 = \begin{cases} \beta J_{m+t} G^*, \\ \beta \sum\limits_{k=-t}^{p} C_k A_{-k} G^*. \end{cases} \tag{6.25}$$

Proof. The validity of (6.25) is a consequence of (6.24). □

Remark 6.4. Note that $J_k G^* = 0$ if and only if $J_k = 0$, since G^* has full rank.

As in the finite dimensional case, let us consider a situation described in the following.

Proposition 6.5. *Let $G^* B_{-t}$ be boundedly invertible and let $m + t = 0$. Then the limit H_0 exists, is independent of the parameter β and*

$$H_0 = C_{-t}(G^* C_{-t})^{-1} G^*. \tag{6.26}$$

Proof. Using Eqs. (6.22) and (6.23) in the identity

$$I_r = [I_r + \beta G^* B(u)^\# F][I_r + \beta G^* B(u)^\# F]^{-1},$$

we obtain

$$I_r = \beta G^* C_{-t} A_t + \mathrm{O}(u) \tag{6.27}$$

or

$$A_t = \frac{1}{\beta}(G^* C_{-t})^{-1}. \tag{6.28}$$

Now, Proposition 6.3 applies and the representation (6.26) results if we use (6.28) in (6.25). □

Now, when H_0 exists, it follows from (6.16) that

$$x_\beta(0;\infty) - x(0;\infty) = H_0 x(0;\infty). \tag{6.29}$$

On the other hand, (6.17) shows that $H(u;0) = 0$ for $u > 0$, so that

$$x_0(u;\infty) - x(0;\infty) = 0 \quad \text{for } u > 0$$

and therefore also

$$x_\beta(u;\infty) - x(0;\infty) \sim 0 \quad \text{for } u > 0 \text{ and } 0 < \beta, \text{ small.} \tag{6.30}$$

Comparing (6.29) and (6.30) we find that $x_\beta(u;\infty)$ undergoes a rapid change for $0 < \beta$ and $u > 0$ if $H_0 x(0;\infty) \neq 0$. We see that this is a *boundary layer phenomenon.*

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